

Nonstandard Poincare and Heisenberg Algebras

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ABSTRACT

New deformations of the Poincare group $Fun(P(1+1))$ and its dual enveloping algebra $U(p(1+1))$ are obtained as a contraction of the h -deformed (Jordanian) quantum group $Fun(SL_h(2))$ and its dual. A non-standard quantization of the Heisenberg algebra $U(h(1))$ is also investigated.

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Off late, considerable interest has been generated towards the nonstandard quantization of Lie groups and algebras, commonly known as h or Jordanian deformation [1-4]. A peculiar feature is that the corresponding universal R matrix is triangular ie $R^{-1} = R_{21}$. Hence it is sometimes also called triangular deformation. These deformations were further extended to the case of supergroups [5]. The contraction method is a useful technique to study inhomogeneous groups. This was employed first by Celeghini et al [11] to obtain quantization of some nonsemisimple groups. Recently attempts have been made to apply it to the Jordanian case [6-8] where the deformation parameter h has a dimension, like the κ deformation [14].

In this letter we propose to obtain a nonstandard quantization of some of the simplest inhomogeneous groups - the (1+1) dimensional Poincare group, its dual enveloping algebra, and the Heisenberg algebra $U(h(1))$ via a contraction of $Fun(SL_h(2))$ and its dual $U_h(sl(2))$. The 3-dim Heisenberg algebra is further extended to 4 dimensions. Another deformation of the 2-dimensional Poincare group can be found in [8,9] which was obtained by simultaneously contracting the deformation parameter h . Our purpose here is to introduce a scaling of the generators in such a way that h remains unscaled.

$Fun(SL_h(2, R))$ is generated by the matrix T

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (1)$$

modulo the relations

$$\begin{aligned} [c, a] &= hc^2, \quad [b, a] = h - ha^2, \quad [a, d] = hac - hdc, \\ [c, d] &= hc^2, \quad [b, d] = h - hd^2, \quad [c, b] = hac + hcd. \end{aligned} \quad (2)$$

The Hopf algebra structure is given by the following co-product, co-unit and antipode (co-inverse)

$$\Delta(T_{ij}) = T_{ik} \otimes T_{kj}, \quad i, j = 1, 2, \quad (3)$$

$$\varepsilon(T_{ij}) = \delta_{ij}, \quad (4)$$

$$S(T) = \begin{pmatrix} d - hc & -b + ha - hd + h^2c \\ -c & a + hc \end{pmatrix} \quad (5)$$

and the determinant

$$D_h = ad - bc - hac = 1 \quad (6)$$

The universal enveloping algebra $U_h(sl(2))$ introduced by Ohn [4] is generated by J_+ , J_- and J_3 with the following Hopf structure

$$[J_3, J_+] = \frac{2 \sinh(hJ_+)}{h}, \quad [J_3, J_-] = -\{J_-, \cosh(hJ_+)\}, \quad [J_+, J_-] = J_3 \quad (7)$$

$$\begin{aligned} \Delta(J_+) &= J_+ \otimes I + I \otimes J_+ \\ \Delta(J_-) &= J_- \otimes e^{hJ_+} + e^{-hJ_+} \otimes J_- \\ \Delta(J_3) &= J_3 \otimes e^{hJ_+} + e^{-hJ_+} \otimes J_3 \end{aligned} \quad (8)$$

$$\varepsilon(J_+) = \varepsilon(J_-) = \varepsilon(J_3) = 0 \quad (9)$$

$$S(J_+) = -J_+, \quad S(J_-) = -e^{hJ_+} J_- e^{-hJ_+}, \quad S(J_3) = -e^{hJ_+} J_3 e^{-hJ_+} \quad (10)$$

The duality between $Fun(SL_h(2))$ and $U_h(sl(2))$ is given by

$$\langle J_+, T \rangle = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \langle J_-, T \rangle = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \langle J_3, T \rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (11)$$

We now apply the following transformation

$$\alpha = a, \quad \beta = b, \quad \gamma = \epsilon^{-1}c, \quad \delta = d \quad (12)$$

Proposition 1 : In the limit $\epsilon \rightarrow 0$ these new generators define an algebra of functions on the nonstandard two dimensional Poincare group $P_h(1+1)$. The corresponding algebra and coalgebra is as follows:

$$\begin{aligned} [\gamma, \alpha] &= 0, & [\beta, \alpha] &= h - h\alpha^2, \\ [\alpha, \delta] &= 0, & [\beta, \delta] &= h - h\delta^2, \\ [\gamma, \delta] &= 0, & [\gamma, \beta] &= h\alpha\gamma + h\gamma\delta, \end{aligned} \quad (13)$$

$$\begin{aligned} \Delta(\alpha) &= \alpha \otimes \alpha, & \Delta(\beta) &= \alpha \otimes \beta + \beta \otimes \delta, \\ \Delta(\gamma) &= \gamma \otimes \alpha + \delta \otimes \gamma, & \Delta(\delta) &= \delta \otimes \delta \end{aligned} \quad (14)$$

$$\varepsilon(\alpha) = \varepsilon(\delta) = 1, \quad \varepsilon(\beta) = \varepsilon(\gamma) = 0 \quad (15)$$

$$S(\alpha) = \delta, \quad S(\beta) = -\beta + h\alpha - h\delta, \quad S(\gamma) = -\gamma, \quad S(\delta) = \alpha \quad (16)$$

and the determinant

$$D_h = \alpha\delta = 1 \quad (17)$$

There are only three independent generators due to the determinant condition. It is interesting that the expressions for coproduct and counit of the nonstandard quantum group $Fun(P_h(1+1))$ coincide with that of the standard Euclidean group $Fun(E_q(2))$ [12].

Now by duality, we scale the generators of the enveloping algebra

$$P_+ = J_+, \quad P_- = \epsilon J_-, \quad K = J_3/2 \quad (18)$$

Proposition 2 : K , P_+ and P_- generate the boosts and the translations along the light-cone and define the universal enveloping algebra $U_h(p(1+1))$ which is isomorphic to the inhomogeneous algebra $U_h(iso(1,1))$.

This new quantum algebra obeys the following relations which can be deduced from those of $(U_h(sl(2)))$ by the above contraction:

$$[K, P_+] = \frac{\sinh(hP_+)}{h}, \quad [K, P_-] = -P_- \cosh(hP_+), \quad [P_+, P_-] = 0 \quad (19)$$

$$\begin{aligned} \Delta(P_+) &= P_+ \otimes I + I \otimes P_+ \\ \Delta(P_-) &= J_- \otimes e^{hP_+} + e^{-hP_+} \otimes P_- \\ \Delta(K) &= K \otimes e^{hP_+} + e^{-hP_+} \otimes K \end{aligned} \quad (20)$$

$$\varepsilon(P_+) = \varepsilon(P_-) = \varepsilon(K) = 0 \quad (21)$$

$$S(P_+) = -P_+, \quad S(P_-) = -P_-, \quad S(K) = -K + \sinh(hP_+) \quad (22)$$

It can be easily checked that the algebras defined in Propostions 1 and 2 satisfy all the Hopf axioms. The above Hopf algebra is invariant under the automorphism $(K, P_+, P_-, h) \rightarrow (K, -P_+, -P_-, -h)$. It was mentioned in [7] that the the non-standard $(1+1)$ Poincare group and its dual are isomorphic to each other as Hopf algebras and one can be obtained from the other by a suitable transformation. It should be pointed out that they are isomorphic only at the algebra level. At the level

of coalgebra they are different since the coproducts are not identical after applying the transformation given in [7].

The Casimir Operator of $U_h(sl(2))$ is [13]

$$C = \frac{J_3^2}{2} + \frac{\sinh(hJ_+)J_-}{h} + \frac{J_- \sinh(hJ_+)}{h} + \frac{\cosh^2(hJ_+)}{2} \quad (23)$$

Applying the contraction (18), and, after a suitable renormalization it leads to the Casimir for $U_h(p(1+1))$

$$C = \frac{2P_- \sinh(hP_+)}{h} \quad (24)$$

One can see that this is central, and, reduces to the casimir operator of the classical 2-dim Pioncare algebra as $h \rightarrow 0$.

The universal R-matrix for $U_h(sl(2))$ is given by [9,10,15]

$$R = \exp \frac{h\Delta J_+}{\sinh(h\Delta J_+)} [J_3 \otimes \sinh(hJ_+) - \sinh(hJ_+) \otimes J_3] \quad (25)$$

In terms of the new generators this becomes

$$R = \exp \frac{h\Delta P_+}{\sinh(h\Delta P_+)} [K \otimes \sinh(hP_+) - \sinh(hP_+) \otimes K] \quad (26)$$

which may be regarded as the universal R-matrix for $U_h(p(1+1))$.

Now we shall obtain a deformation of the 3 dimensional Heisenberg algebra $U(h(1))$ again by contracting the algebra of $U_h(sl(2))$ but now with a different scaling. Define new generators

$$A = J_+, \quad A^+ = \epsilon J_-, \quad H = \epsilon J_3 \quad (27)$$

which satisfy the following commutation relations

$$[H, A] = 0, \quad [H, A^+] = 0, \quad [A, A^+] = H \quad (28)$$

Proposition 3 : A (annihilation operator), A^+ (creation operator) and H (cartan) span a 3-dim deformed heisenberg algebra $U(h_h(1))$.

Note that the above algebra coincides with its classical counterpart. However the quantum nature is manifested in the following coalgebra structure:

$$\begin{aligned} \Delta(A) &= A \otimes I + I \otimes A \\ \Delta(A^+) &= A^+ \otimes e^{hA} + e^{-hA} \otimes A^+ \\ \Delta(H) &= H \otimes e^{hA} + e^{-hA} \otimes H \end{aligned} \quad (29)$$

$$\varepsilon(A) = \varepsilon(A^+) = \varepsilon(H) = 0 \quad (30)$$

$$S(A) = -A, \quad S(A^+) = -e^{hA} A_+ e^{-hA}, \quad S(H) = -H \quad (31)$$

In the q case the generators A and A^+ were taken to be hermitian conjugate of each other contrary to the present situation. The reason for this is that there is no $*$ structure defined on the Jordanian quantum group and hence does not have a compact form.

We now pass on to investigate the case of 4-dimensional heisenberg algebra. This is done by central extension i.e. $sl_h(2) \otimes u(1)$. Introduce K as the new $u(1)$ generator and apply the transformation

$$A = J_+, \quad A^+ = \epsilon J_-, \quad N = -\frac{J_3}{2} + \frac{H}{2\epsilon}, \quad H = K \quad (32)$$

Proposition 4 : The above transformation in the singular limit $\epsilon \rightarrow 0$ yields a nonstandard quantization of the 4-dim heisenberg algebra which is not a Hopf algebra.

The deformed algebra is

$$[A, A^+] = H, \quad [N, A] = -\frac{\sinh(hA)}{h}, \quad [N, A^+] = \frac{1}{2}\{A^+, \cosh(hA)\}, \quad [H, \cdot] = 0 \quad (33)$$

The coproduct, counit and antipode are given by

$$\begin{aligned} \Delta(A) &= A \otimes I + I \otimes A \\ \Delta(A^+) &= A^+ \otimes e^{hA} + e^{-hA} \otimes A^+ \\ \Delta(H) &= H \otimes e^{hA} + e^{-hA} \otimes H \\ \Delta(N) &= N \otimes e^{hA} + e^{-hA} \otimes N \end{aligned} \quad (34)$$

$$\varepsilon(A) = \varepsilon(A^+) = \varepsilon(H) = \varepsilon(N) = 0 \quad (35)$$

$$S(A) = -A, \quad S(A^+) = -e^{hA} A_+ e^{-hA}, \quad S(H) = -H, \quad S(N) = -e^{hA} N e^{-hA} \quad (36)$$

Note that however, the generators A and N form a Hopf subalgebra of the above. Infact it turns out that it might not be possible to obtain an h Hopf deformation of the above oscillator algebra in the framework of the contraction technique.

We remark in passing that it is not straightforward to obtain the universal R-matrix for the heisenberg algebra by contracting (25), as it does not involve central

factors which could be neglected even if they were singular (unlike the case for q deformation [11]). It would be interesting to obtain an expression for such an R-matrix.

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Note added: After the submission of this work, the paper [16] was brought to my attention where the nonstandard Hopf deformation of the 4-dim oscillator algebra was obtained by a different method.

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